

THE CONVOLUTION ALGEBRA STRUCTURE ON $K^G(\mathcal{B} \times \mathcal{B})$

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ABSTRACT. We show that the convolution algebra $K^G(\mathcal{B} \times \mathcal{B})$ is isomorphic to the Based ring of the lowest two-sided cell of the extended affine Weyl group associated to G , where G is a connected reductive algebraic group over the field \mathbb{C} of complex numbers and \mathcal{B} is the flag variety of G .

INTRODUCTION

We are interested in understanding the equivariant group $K^G(\mathcal{B} \times \mathcal{B})$, where G is a connected reductive algebraic group over \mathbb{C} and \mathcal{B} is the flag variety of G .

When G has simply connected derived subgroup, the Künneth formula $K^G(\mathcal{B} \times \mathcal{B}) \simeq K^G(\mathcal{B}) \otimes_{R_G} K^G(\mathcal{B})$ is proved in Proposition 1.6 of [KL] and plays an important role in Kazhdan-Lusztig's proof of Deligne-Langlands conjecture for affine Hecke algebra associated to G , where R_G denotes the representation ring of G . Furthermore, by Theorem 1.10 of [Xi], the convolution algebra structure on $K^G(\mathcal{B} \times \mathcal{B})$ is isomorphic to the based ring of the lowest two-sided cell of the extended affine Weyl group associated to G .

In general, $K^G(\mathcal{B} \times \mathcal{B})$ is not isomorphic to $K^G(\mathcal{B}) \otimes_{R_G} K^G(\mathcal{B})$. To set a Deligne-Langlands-Lusztig classification for affine Hecke algebra associated to G , it seems useful to understand the equivariant K -groups $K^G(\mathcal{B} \times \mathcal{B})$. The main result of this paper is Theorem 1.1, which says that the convolution algebra on $K^G(\mathcal{B} \times \mathcal{B})$ is isomorphic to the based ring of the lowest two-sided cell of the extended affine Weyl group associated to G . Since the based ring is known explicitly in [Xi], the main result gives an explicit description to the equivariant K -group $K^G(\mathcal{B} \times \mathcal{B})$.

1. PRELIMINARY

1.1. Let G be a connected reductive algebraic group over \mathbb{C} , B a Borel subgroup of G and T a maximal torus of G , such that $T \subset B$. The Weyl group $W_0 = N_G(T)/T$ of G acts on the character group $X = \text{Hom}(T, \mathbb{C}^*)$ of T . Using this action we define the extended affine Weyl group $W = X \rtimes W_0$.

By classification theorem for connected reductive algebraic groups, there exists a connected reductive algebraic group \tilde{G} with simply connected derived subgroup such that G is a quotient group of \tilde{G} modulo a finite subgroup of the center of \tilde{G} . Denote by $\pi : \tilde{G} \rightarrow G$ the quotient homomorphism. Set $\tilde{B} = \pi^{-1}(B)$, $\tilde{T} = \pi^{-1}(T)$, $\tilde{X} = \text{Hom}(\tilde{T}, \mathbb{C}^*)$ and $\tilde{W} = \tilde{X} \rtimes W_0$. Note that X is naturally a subgroup of \tilde{X} of finite index, hence W is a naturally subgroup of \tilde{W} of finite index.

Let $R \subset X$ be the root of G and \tilde{G} . Let $R^- \subset R$ to be the set of negative roots determined by B . Set $R^+ = R - R^-$. Let $\Delta \subset R^+$ be set of simple positive roots.

Denote by λ_α the dominant fundamental weight corresponding to a simple positive root $\alpha \in R^+$. For any $w \in W_0$, define $x_w = w^{-1}(\prod_{\alpha \in \Delta, w^{-1}(\alpha) < 0} \lambda_\alpha) \in \tilde{X}$. It is known that $\mathbb{Z}[\tilde{X}]$ is a free $\mathbb{Z}[X]^{W_0}$ -module with a basis $\{x_w | w \in W_0\}$.

Let $\ell : \tilde{W} \rightarrow \mathbb{N}$ be the length function. Note that $\ell(w\lambda) = \ell(w) + \ell(\lambda)$ for any $w \in W_0$ and any dominant weight $\lambda \in \tilde{X}$. Also we have $\ell(\lambda_\alpha s_\alpha) = \ell(\lambda_\alpha) - 1$ for any positive simple root $\alpha \in \Delta$.

1.2. Let $\Sigma = \{wx_w | w \in W_0\}$. Then the lowest two-sided cell \tilde{c}_0 of \tilde{W} consists of elements $f^{-1}w_0\chi g$ with $f, g \in \Sigma$ and $\chi \in \tilde{X}^+$. (See [Shi]) Here w_0 is the longest element of W_0 and \tilde{X}^+ is the set of dominant weights of \tilde{X} . The lowest two-sided cell of W is $c_0 = \tilde{c}_0 \cap W$. The ring structure of $J_{\tilde{c}_0}$ of \tilde{c}_0 is defined in §2 of [L1] and explicitly determined in §4 [Xi]. As a \mathbb{Z} -module, it is free with a basis $t_z, z \in \tilde{c}_0$. The based ring J_{c_0} of c_0 is a subring of $J_{\tilde{c}_0}$ spanned by all $t_z, z \in c_0$.

1.3. For an algebraic group M over \mathbb{C} and an variety Z over \mathbb{C} which admits an algebraic action of M , denote by $K^M(Z)$ the Grothendieck group of M -equivariant coherent sheaves on Z . We refer to Chapter 5 of [CG] for more about the equivariant K -group $K^M(Z)$.

There is a natural map $L : \mathbb{Z}[\tilde{X}] \rightarrow K^{\tilde{G}}(\mathcal{B})$ which associates $\chi \in \tilde{X}$ to the unique equivariant line bundle $[L(\chi)]$ on \mathcal{B} such that \tilde{T} acts on the fibre $L(\chi)_B$ over $B \in \mathcal{B}$ via χ . Here $\mathcal{B} = \tilde{G}/\tilde{B} = G/B$ is the flag variety. It is well known that L is an isomorphism. By abuse of notation, we will use χ and $[L(\chi)]$ interchangeably in the following context.

The convolution on $K^G(\mathcal{B} \times \mathcal{B})$ is defined by

$$\mathfrak{F} * \mathfrak{F}' = Rp_{13*}(p_{12}^*\mathfrak{F} \bigotimes_{\mathcal{O}_{\mathcal{B} \times \mathcal{B} \times \mathcal{B}}}^L p_{23}^*\mathfrak{F}'),$$

where $\mathfrak{F}, \mathfrak{F}' \in K^G(\mathcal{B} \times \mathcal{B})$ and $p_{12}, p_{13}, p_{23} : \mathcal{B} \times \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B} \times \mathcal{B}$ are obvious natural projections. Identifying $K^{\tilde{G}}(\mathcal{B} \times \mathcal{B})$ with $K^{\tilde{G}}(\mathcal{B}) \otimes_{R_{\tilde{G}}} K^{\tilde{G}}(\mathcal{B}) \simeq \mathbb{Z}[\tilde{X}] \otimes_{\mathbb{Z}[\tilde{X}]^{W_0}} \mathbb{Z}[\tilde{X}]$, the convolution becomes

$$(\chi_1 \otimes \chi_2) * (\chi'_1 \otimes \chi'_2) = (\chi_2, \chi'_1)\chi_1 \otimes \chi'_2,$$

where $(,) : \mathbb{Z}[\tilde{X}] \otimes_{\mathbb{Z}[\tilde{X}]^{W_0}} \mathbb{Z}[\tilde{X}] \rightarrow \mathbb{Z}[\tilde{X}]^{W_0}$ is given by

$$(\chi_2, \chi'_1) = \delta^{-1} \sum_{w \in W_0} (-1)^{\ell(w)} w(\chi_2 \rho \chi'_1).$$

Here $\delta = \prod_{\alpha \in R^+} (\alpha^{\frac{1}{2}} - \alpha^{-\frac{1}{2}})$ and $\rho = \prod_{\alpha \in R^+} \alpha^{\frac{1}{2}}$.

For $f = wx_w \in \Sigma$, set $x_f = x_w$. Since $(,)$ is a perfect pairing (See Proposition 1.6 in [KL]), we can find $y_f \in \mathbb{Z}[\tilde{X}]$ such that $(x_f, y_f) = \delta_{f, f'}$. The following result is due to N. Xi. (See Theorem 1.10 in [Xi].)

(*) The map $\sigma : J_{\tilde{c}_0} \rightarrow K^{\tilde{G}}(\mathcal{B} \times \mathcal{B}) \simeq \mathbb{Z}[\tilde{X}] \otimes_{\mathbb{Z}[\tilde{X}]^{W_0}} \mathbb{Z}[\tilde{X}]$ given by $t_{f^{-1}w_0\chi f'} \mapsto V(\chi)y_f \otimes x_{f'}$ for $\chi \in \tilde{X}^+$ and $f, f' \in \Sigma$ is an isomorphism of $R_{\tilde{G}}$ -algebras. Here $V(\chi) \in R_{\tilde{G}}$ stands for the irreducible \tilde{G} -module with highest weight χ .

Now we state the main result of this paper.

Theorem 1.1. (a) The natural map $i : K^G(\mathcal{B} \times \mathcal{B}) \rightarrow K^{\tilde{G}}(\mathcal{B} \times \mathcal{B})$ is an injective of homomorphism of algebra.
 (b) As a \mathbb{Z} -module, the image of i is spanned by $\{V(\chi)y_f \otimes x_{f'}; \chi \in \tilde{X}^+, f, f' \in \Sigma, f^{-1}w_0\chi f' \in W\}$.
 (c) In particular, via the isomorphism σ in (*), $J_{\tilde{c}_0}$ is isomorphic to the convolution algebra $K^G(\mathcal{B} \times \mathcal{B})$ as R_G -algebras.

2. PROOF OF THEOREM 1.1

2.1. Set $\Omega = \tilde{W}/W = \tilde{X}/X = \{\lambda X; \lambda \in \tilde{X}\}$, which is a finite abelian group. For a left coset λX , let $\mathbb{Z}[\lambda X]$ be the \mathbb{Z} -submodule of the group algebra $\mathbb{Z}[\tilde{X}]$ spanned by elements in λX . For any $A \in \mathbb{Z}[\lambda X]$ and $B \in \mathbb{Z}[\mu X]$, we have $AB \in \mathbb{Z}[\lambda\mu X]$. Moreover if there is $C \in \mathbb{Z}[\tilde{X}]$ such that $A = BC$, then $C \in \mathbb{Z}[\lambda\mu^{-1}X]$.

Lemma 2.1. For $f \in \Sigma$, we have $y_f \in \mathbb{Z}[x_f^{-1}X]$.

Proof. For $f, f' \in \Sigma$, set $A_{f, f'} = (x_f, x_{f'}) = \delta^{-1} \sum_{w \in W_0} (-1)^{\ell(w)} w(x_f \rho x_{f'})$ which lies in $\mathbb{Z}[x_f x_{f'} X]$. Let $(A^{f, f'})_{(f, f') \in \Sigma \times \Sigma}$ be the inverse matrix of $(A_{f, f'})_{(f, f') \in \Sigma \times \Sigma}$. Then a direct computation shows $A^{f, f'} \in \mathbb{Z}[x_f^{-1} x_{f'}^{-1} X]$. Since $y_f = \sum_{f' \in \Sigma} A^{f, f'} x_{f'}$, we have $y_f \in \mathbb{Z}[x_f^{-1} X]$. \square

2.2. For $w \in W_0$, let $Y_w \subset \mathcal{B} \times \mathcal{B}$ be the G -orbit containing (B, wB) . Then $\mathcal{B} \times \mathcal{B} = \coprod_{w \in W_0} Y_w$ and the projection to the first factor $p_1 : Y_w \rightarrow \mathcal{B}$ is an affine bundle of rank $\ell(w)$. Numbering the elements of W_0 as u_1, u_2, \dots, u_r such that $u_i \not\prec u_j$ if $j < i$. Let $F_i = \coprod_{j \leq i} Y_{u_j}$. Then F_i is closed in $\mathcal{B} \times \mathcal{B}$. We have the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & K^G(F_{i-1}) & \longrightarrow & K^G(F_i) & \longrightarrow & K^G(Y_{u_i}) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & K^{\tilde{G}}(F_{i-1}) & \longrightarrow & K^{\tilde{G}}(F_i) & \longrightarrow & K^{\tilde{G}}(Y_{u_i}) \longrightarrow 0
\end{array}$$

where all the morphisms are natural and obvious. By [CG], the rows of the diagram are exact sequences. Since $p_1 : Y_{u_i} \rightarrow \mathcal{B}$ is an affine bundle, we have the following commutative diagram

$$\begin{array}{ccccc}
K^G(Y_{u_i}) & \xrightarrow{\sim} & K^G(\mathcal{B}) & \longrightarrow & K^{\tilde{G}}(\mathcal{B}) \xrightarrow{\sim} K^{\tilde{G}}(Y_{u_i}) \\
& & \downarrow \wr & & \downarrow \wr \\
& & \mathbb{Z}[X] & \longrightarrow & \mathbb{Z}[\tilde{X}]
\end{array}$$

which shows that the natural morphism $K^G(Y_{u_i}) \rightarrow K^{\tilde{G}}(Y_{u_i})$ is injective. Using induction on i , we see that the natural morphism $i : K^G(\mathcal{B} \times \mathcal{B}) \rightarrow K^{\tilde{G}}(\mathcal{B} \times \mathcal{B})$ is injective. One shows directly that i is a homomorphism of convolution algebras. Part (a) of Theorem 1.1 is proved.

2.3. For $w \in W_0$, let $X_w = \tilde{B}w\tilde{B}/\tilde{B}$ and $X^w = wB^+\tilde{B}/\tilde{B}$, where $\tilde{B}^+ \supset \tilde{T}$ is the opposite of \tilde{B} . Then X^w is an \tilde{T} -invariant open neighborhood of X_w in \mathcal{B} . Set $X_i = \coprod_{j \leq i} X_{u_j}$. Note that $\mathcal{B} \times \mathcal{B} = \tilde{B} \backslash (\tilde{G} \times \mathcal{B})$, where the action of \tilde{B} on $\tilde{G} \times \mathcal{B}$ is given by $b(g, h\tilde{B}) = (gb^{-1}, bh\tilde{B})$. Hence $K^{\tilde{G}}(\mathcal{B} \times \mathcal{B}) \simeq K^{\tilde{B}}(\mathcal{B}) \simeq K^{\tilde{T}}(\mathcal{B})$. Similarly, $K^{\tilde{G}}(F_i) \simeq K^{\tilde{T}}(X_i)$ and $K^{\tilde{G}}(Y_w) \simeq K^{\tilde{T}}(X_w)$.

Let $j_w : Y_w \rightarrow \mathcal{B} \times \mathcal{B}$ be the natural G -equivariant inclusion. Since X_w is a \tilde{T} -equivariant vector bundle over a single point. We have $K^{\tilde{G}}(Y_w) \simeq K^{\tilde{T}}(X_w) \simeq \mathbb{Z}[\tilde{X}]$. Then a direct computation shows that the induced homomorphism Lj_w^* of equivariant K -groups is given by

$$\begin{aligned}
Lj_w^* : K^{\tilde{G}}(\mathcal{B} \times \mathcal{B}) &\rightarrow K^{\tilde{G}}(Y_w) \simeq \mathbb{Z}[\tilde{X}], \\
x_1 \otimes x_2 &\mapsto x_1 w(x_2),
\end{aligned}$$

where $x_1 \otimes x_2 \in \mathbb{Z}[\tilde{X}] \otimes_{\mathbb{Z}[\tilde{X}]^{W_0}} \mathbb{Z}[\tilde{X}] \simeq K^{\tilde{G}}(\mathcal{B} \times \mathcal{B})$.

Proposition 2.2. *Let $l \in K^{\tilde{G}}(\mathcal{B} \times \mathcal{B})$, then $l \in K^G(\mathcal{B} \times \mathcal{B})$ if and only if $Lj_w^*(l) \in K^G(Y_w)$ for any $w \in W_0$.*

Proof. Denote by $k_w : X_w \hookrightarrow \mathcal{B}$, $k_i : X_i \hookrightarrow \mathcal{B}$ and $j_i : F_i \hookrightarrow \mathcal{B} \times \mathcal{B}$ the natural immersions. Then we have

$$\begin{array}{ccccc}
K^{\tilde{G}}(\mathcal{B} \times \mathcal{B}) & \xrightarrow{Lj_i^*} & K^{\tilde{G}}(F_i) & \xrightarrow{\text{res}} & K^{\tilde{G}}(Y_{u_i}) \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
K^{\tilde{T}}(\mathcal{B}) & \xrightarrow{Lk_i^*} & K^{\tilde{T}}(X_i) & \xrightarrow{\text{res}} & K^{\tilde{T}}(X_{u_i})
\end{array}$$

Hence our proposition is equivalent to the following statement

For any $l \in K^{\tilde{T}}(\mathcal{B})$, $l \in K^T(\mathcal{B})$ if and only if $Lk_w^(l) \in K^T(X_w)$ for any $w \in W_0$.*

The "only if" part is obviously. We show the "if" part. Note that the support $\text{supp}(l)$ of l belongs to some X_i , we argue by induction on i . If $\text{supp}(l) = \emptyset$, that is, $l = 0$, then the statement follows trivially. Suppose the proposition holds for any element in $K^{\tilde{T}}(\mathcal{B})$ whose support belongs to X_{i-1} . We show it also holds for $z \in K^{\tilde{T}}(\mathcal{B})$ with $\text{supp}(l) \subset X_i$.

Since we have the following \tilde{T} -equivariant morphism

$$\begin{array}{ccc} X_{u_i} & \xrightarrow{\quad} & X^{u_i} \\ \downarrow \wr & & \downarrow \wr \\ \bigoplus_{\alpha \in R^-, u_i^{-1}(\alpha) > 0} \mathbb{C}_\alpha & \longrightarrow & \bigoplus_{\beta \in R^+} \mathbb{C}_{u_i(\beta)}, \end{array}$$

where \mathbb{C}_α denotes the one dimensional vector space \mathbb{C} on which \tilde{T} acts via the character α . By Proposition 5.4.10 in [CG],

$$Lk_{u_i}^*(l) = \prod_{\alpha \in R^+, u_i^{-1}(\alpha) > 0} (1 - \alpha^{-1}) l|_{X_{u_i}} \in K^T(X_{u_i}).$$

Since $\prod_{\alpha \in R^+, u_i^{-1}(\alpha) > 0} (1 - \alpha^{-1}) \in K^T(X_{u_i})$, then $l|_{X_{u_i}} \in K^T(X_{u_i})$ by 2.1. Since X_{u_i} is a \tilde{T} -stable open subset of X_i , there exists $l' \in K^T(X_i)$ such that $l'|_{X_{u_i}} = l|_{X_{u_i}}$. Then $\text{supp}(l - l') \subset X_{i-1}$. Using induction hypothesis, we have $l - l' \in K^T(\mathcal{B})$. Hence $l = (l - l') + l' \in K^T(\mathcal{B})$ and the proof is finished. \square

Corollary 2.3. *Let $z = f^{-1}w_0\chi f'$ with $f, f' \in \Sigma$ and $\chi \in \tilde{X}^+$. Then $z \in W$ if and only if $\sigma(z) = V(\chi)y_f \otimes x_{f'} \in K^G(\mathcal{B} \times \mathcal{B})$.*

Proof. Obviously $z \in W$ if and only if $x_f^{-1}\chi x_{f'} \in X$. On the other hand, $V(\chi) \in \mathbb{Z}[\chi X]$. By Lemma 2.1, $y_f \in \mathbb{Z}[x_f^{-1}X]$. Then by Proposition 2.2, $V(\chi)y_f \otimes x_{f'} \in K^G(\mathcal{B} \times \mathcal{B})$ if and only if $V(\chi)y_f x_{f'} \in \mathbb{Z}[X]$, which is equivalent to $x_f^{-1}\chi x_{f'} \in X$. \square

Proof of part (b) and (c) of Theorem 1.1. By Corollary 2.3, $\sigma(J_0) \subset K^G(\mathcal{B} \times \mathcal{B})$. It remains to show that $K^G(\mathcal{B} \times \mathcal{B}) \subset \sigma(J_0)$. Let $l \in K^G(\mathcal{B} \times \mathcal{B})$. Due to (*), we can assume that

$$l = \sum_{f, f' \in \Sigma} a_{f, f'} V(\chi_{f, f'}) y_f \otimes x_{f'}$$

with $a_{f, f'} \in \mathbb{Z}$, $\chi_{f, f'} \in \tilde{X}^+$. Since $y_f \otimes x_{f'} = \sigma(t_{f^{-1}w_0f}) \in K^G(\mathcal{B} \times \mathcal{B})$ for each $f \in \Sigma$, we have

$$(y_f \otimes x_{f'}) * l * (y_{f'} \otimes x_{f'}) = a_{f, f'} V(\chi_{f, f'}) y_f \otimes x_{f'} \in K^G(\mathcal{B} \times \mathcal{B}).$$

Hence by Corollary 2.3, $f^{-1}w_0\chi_{f,f'}f' \in J_0$ whenever $a_{f,f'} \neq 0$. Hence $l = \sum_{f,f'} a_{f,f'} \sigma(t_{f^{-1}w_0\chi_{f,f'}}) \in \sigma(J_0)$. \square

3. SOME RESULTS ON $K^G(\mathcal{P} \times \mathcal{P})$

3.1. Let $I \subset \Delta$ be a subset and P the parabolic subgroup of type I containing B . Define $\mathcal{P} = G/P$ be the variety of all parabolic subgroups of type I .

Let \mathcal{D} be the set of double cosets of W_0 with respect to $W_I \subset W_0$. Here W_I is the parabolic subgroup generated by I . For each $w \in W_0$, define

$$Z_{\bar{w}} = \{(P, P') \in \mathcal{P} \times \mathcal{P} \mid (P, P') \text{ is conjugate to } (P, {}^wP)\},$$

where \bar{w} denotes the double coset $W_I w W_I$. Then we have

$$\mathcal{P} \times \mathcal{P} = \coprod_{d \in \mathcal{D}} Z_d.$$

For any double coset $d \in \mathcal{D}$, there is a unique element $u_d \in d$ such that u_d is the smallest in d under the Bruhat order. Let $d, d' \in \mathcal{D}$, we say $d \geq d'$ if $u_d \geq u_{d'}$ under the Bruhat order.

Lemma 3.1. *With notations as above, then we have $d \geq d'$ if and only if $\bar{Z}_d \supset Z_{d'}$.*

Proof. Consider the natural projections $Y_{u_d} \rightarrow Z_d$ and $Y_{u_{d'}} \rightarrow Z_{d'}$, which are restrictions of the natural projection $p : \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{P} \times \mathcal{P}$ to Y_{u_d} and $Y_{u_{d'}}$, respectively. Since $u_d \geq u_{d'}$, we have $\bar{Y}_{u_d} \supset Y_{u_{d'}}$. Hence $Z_{d'} \subset p(\bar{Y}_{u_{d'}}) \subset p(\bar{Y}_{u_d}) = \bar{Z}_d$. The “if part” follows from the fact that the morphism p above is projective. \square

Proposition 3.2. *Let $d \in \mathcal{D}$. We have the following short exact sequence:*

$$0 \rightarrow K^G(Z_{<d}) \rightarrow K^G(\bar{Z}_d) \rightarrow K^G(Z_d) \rightarrow 0.$$

Here $Z_{<d} = \bar{Z}_d - Z_d$.

Proof. It suffices to prove the injection $K^G(Z_{<d}) \hookrightarrow K^G(\bar{Z}_d)$. Note that we have a natural injective morphism $K^G(Z) \hookrightarrow K^T(Z)$ for $Z = Z_{<d}$ or $Z = \bar{Z}_d$, where $T \subset G$ is a maximal torus. Hence it suffices to prove the result for torus T . By Lemma 1.6 in [L2], we just have to show that $K^T(Z_d)$ is a free R_T -module for all $d \in \mathcal{D}$. Let's consider the projection $q : Z_d \rightarrow \mathcal{P}$ given by $(P, P') \mapsto P$. Let $x \in \{w \in W_0 \mid w \text{ is of minimal length among } wW_I\}$. Define $Z_d^x = q^{-1}(BxP/P)$ which is T -stable. Thus it suffices to show $K^T(Z_d^x)$ is a free R_T -module. Note that $Z_d^x = B_x \backslash (B \times F_x)$, where $B_x = \{b \in B; {}^{bx}P = {}^xP\}$ and $F_x = q^{-1}({}^xP)$. Then we have

$$K^T(Z_d^x) = K^T(B_x \backslash (B \times F_x)) = K^B(B_x \backslash (B \times F_x)) = K^T(F_x).$$

Since $F_x = {}^xP/({}^xP \cap {}^{xu_d}P)$, and ${}^xP/({}^xP \cap {}^{xu_d}P)$ contains a Borel subgroup of a Levy subgroup of xP containing T , it admits a partition which are T -vector spaces. Hence the proposition follows. \square

3.2. Let $w \in d = W_I w W_I$. Choose Borel subgroups $\hat{B}_w \subset P$ and $\hat{B}'_w \subset {}^wP$ such that $\hat{B}_w \cap \hat{B}'_w$ is a Borel subgroup of $P \cap {}^wP$. Assume $(\hat{B}_w, \hat{B}'_w) \in Y_u$ for some $u \in d$. Consider the natural projection $p|_{Y_u} : Y_u \rightarrow Z_{\bar{w}}$. It is easy to see $p^{-1}(P, {}^wP) = P \cap {}^wP/(\hat{B}_w \cap \hat{B}'_w) = \mathcal{B}_{P \cap {}^wP}$. Here $\mathcal{B}_{P \cap {}^wP}$ denotes the flag variety of $P \cap {}^wP$. Hence $p|_{Y_u}$ is a projective morphism. So $u = u_d$ is the minimal length element in the double coset d .

Define

$$Rp_* : K^G(Y_{u_d}) \longrightarrow K^G(Z_{\bar{w}})$$

$$[\mathcal{F}] \mapsto \sum_i (-1)^i [R^i p_* \mathcal{F}].$$

Let $\chi \in X = \text{Hom}(T, \mathbb{C}^*)$. Denote by θ_χ the G -equivariant line bundle over Y_{u_d} such that T acts on the fiber of θ_χ over $(\hat{B}_w, \hat{B}'_w) \in Y_{u_d}$ via χ .

Proposition 3.3. *With notations in 3.2. The morphism $R(p|_{Y_{u_d}})_* : K^G(Y_{u_d}) \longrightarrow K^G(Z_{\bar{w}})$ defined above is surjective.*

Proof. Let's compute $R(p|_{Y_{u_d}})_*([\theta_\chi])$. Note that p is smooth and projective, and $Z_{\bar{w}}$ is integral (as a scheme). Hence by Corollary 12.9 of [Har], we have $R^i p|_{Y_{u_d}*}(\theta_\chi)$ is vector bundle over $Z_{\bar{w}}$ and $R^i p_*(\theta_\chi)|_{(P, {}^wP)} = H^i(p^{-1}(P, {}^wP), \theta_\chi|_{p^{-1}(P, {}^wP)})$ for all $i \geq 0$. Hence when χ is a dominant weight, $Rp_*(\theta_\chi)|_{(P, {}^wP)} = V_\chi$, where V_χ is the irreducible $P \cap {}^wP$ -module with highest weight χ . Note that all $[V_\chi]$ with χ dominant generates $K^{P \cap {}^wP}(\text{pt}) = K^G(Z_{\bar{w}})$. Hence Rp_* is surjective. \square

Corollary 3.4. *The natural morphism $Rp_* : K^G(\mathcal{B} \times \mathcal{B}) \rightarrow K^G(\mathcal{P} \times \mathcal{P})$ is surjective.*

Proof. Let $l \in K^G(\mathcal{P} \times \mathcal{P})$. We show that l lies in the image of Rp_* by induction on the dimension of the support of $\text{supp}(l)$. If $\text{supp}(l) = \emptyset$, hence $l = 0$, it follows trivially. Now we assume that the statement holds for any l' such that $\text{supp}(l') \subset Z_{<d}$ for some $d \in \mathcal{D}$ and any $l' \in K^G(\mathcal{P} \times \mathcal{P})$. Assume $\text{supp}(l) \subset Z_{\leq d}$. Let $l_d = i_d^*(l) \in K^G(Z_d)$, where $i_d : Z_d \hookrightarrow Z_{\leq d}$ be the natural open immersion. By Proposition 3.3, there is $f_d \in K^G(Y_{u_d})$ such that $R(p|_{Y_{u_d}})_*(f_d) = l_d$. Now extend f_d to some $\bar{f}_d \in K^G(\bar{Y}_{u_d})$. Thanks to 3.2, we have $p^{-1}(Z_d) \cap \bar{Y}_{u_d} = Y_{u_d}$. Hence $Rp_*(\bar{f}_d)|_{Z_d} = l_d$. Then the support $i_d^*(l - Rp_*(\bar{f}_d)) = 0$. By Proposition 3.2, we have $\text{supp}(l - Rp_*(\bar{f}_d)) \subset Z_{<d}$. By induction hypothesis, the statement follows. \square

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